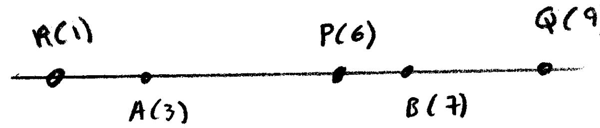
P124

[1] $AO = 4$, $AB = |-4+2| = 2$, $BC = |-2-5| = 7$

P125

[2] $A(3), B(7)$. P div AB 3:1 Internal.



$$P = \frac{3(7)+(3)}{4} = 6$$

Q external 3:1

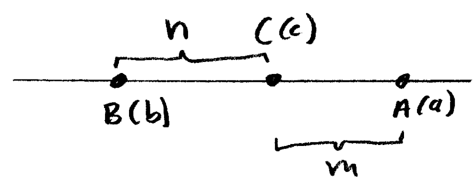
$$q = \frac{3(7)-(3)}{3-1} = \frac{18}{2} = 9$$

R external $m:n = 1:3$

$$r = \frac{7-(3)(3)}{1-3} = \frac{-2}{-2} = 1$$

P126

[3] Prove $c = \dfrac{mb+na}{m+n}$, for $a > b$



$$\frac{m}{n} = \frac{|a-c|}{|b-c|} = \frac{a-c}{c-b}$$

$$mc - mb = na - nc$$

$$mc + nc = mb + na$$

$$\therefore \quad c = \frac{mb+na}{m+n}$$

$\square$

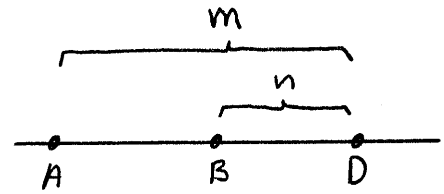
[4.1] $A(-5), B(7)$

$$\bar{x} = \frac{-5+7}{2} = \frac{2}{2} = 1$$

[4.2] $m/n = 3/2$

$$x = \frac{3(7)+(2)(-5)}{5} = \frac{11}{5}$$

[5] Let $A(a)$, $B(b)$ given. Suppose $D(d)$ divides Externally AB in ratio $m:n$.



<u>case</u> $a < b < d$

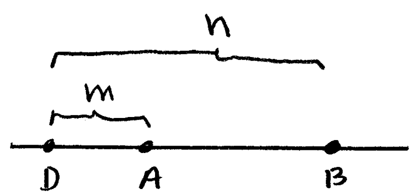
$$\frac{m}{n} = \frac{|AD|}{|BD|}$$

$$\frac{m}{n} = \frac{d-a}{d-b}$$

$$\equiv \quad md - mb = nd - na$$

$$(m-n)d = mb - na$$

$$d = \frac{mb-na}{m-n}$$



<u>case</u> $d < a < b$

$$\frac{m}{n} = \frac{|AD|}{|BD|}$$

$$= \frac{a-d}{b-d}$$

$$\equiv \quad mb - md = na - nd$$

$$(m-n)d = mb - na$$

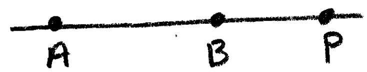
$$d = \frac{mb-na}{m-n}$$

$$\therefore \quad d = \frac{mb-na}{m-n}$$

P126, ctd

[6.1] $A(1), B(5)$

EXT. $m/n = 3/2$



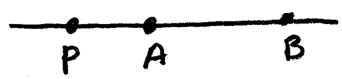
P right of B
$\therefore$ $m > n$

$$P = \frac{3(5) - (2)(1)}{3 - 2}$$

$$= \frac{13}{1} = 13$$

$\therefore$ $P = 13$

[6.2] $m/n = 1/4$



P left of A
$\therefore$ $m < n$

$$P = \frac{(1)(5) - (4)(1)}{1 - 4}$$

$$= \frac{1}{-3}$$

$\therefore$ $P = -\frac{1}{3}$

CHECK

$AP = \frac{4}{3}$

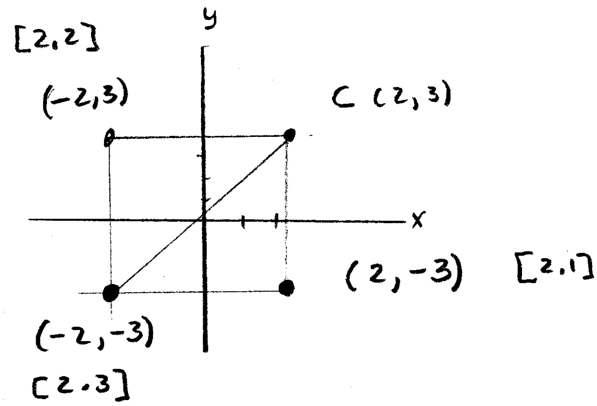
$BP = \frac{16}{3}$

$\frac{AP}{BP} = \frac{\frac{4}{3}}{\frac{16}{3}} = \frac{1}{4}$ ✓

P127

[1] A in II, B in III, C in IV, D in I

[2]



P128

[3] AB // x-axis $\Rightarrow A(x_1, 0), B(x_2, 0)$

Then
$$AB = \sqrt{(x_2-x_1)^2 + (0-0)^2}$$
$$= \sqrt{(x_2-x_1)^2}$$
$$= |x_2-x_1|$$
$$= \text{distance } AB$$

AB// y-axis $\Rightarrow A(0, y_1), B(0, y_2)$

Then
$$AB = \sqrt{(0-0)^2 + (y_2-y_1)^2}$$
$$= |y_2-y_1|$$
$$= \text{distance } AB$$

$\square$

P129

[4,1] $d = \sqrt{(4-0)^2+(-3-0)^2} = \sqrt{25} = 5$

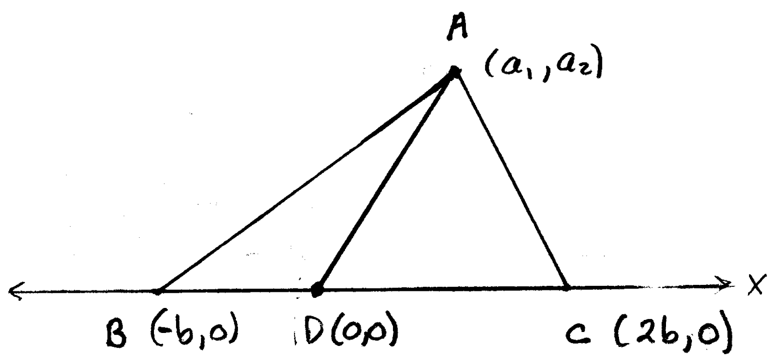
[4,2] $d = \sqrt{(5-3)^2+(0+1)^2} = \sqrt{4+1} = \sqrt{5}$

[4,3] $d = \sqrt{(-3+5)^2+(2+3)^2} = \sqrt{4+25} = \sqrt{29}$

[4,4] $d = \sqrt{(3-3)^2+(-19+2)^2} = \sqrt{289} = 17$

[5] $ D div BC of ΔABC int. ratio 1:2.

Prove $2AB^2 + AC^2 = 3(AD^2 + 2BD^2)$



$AB^2 = (a_1+b)^2 + (a_2-0)^2$

$= a_1^2 + 2a_1b + b^2 + a_2^2$

$AC^2 = (a_1+2b)^2 + a_2^2$

$= a_1^2 + 4a_1b + 4b^2 + a_2^2$

$LHS = 2AB^2 + AC^2$

$= (2a_1^2 + 4a_1b + 2b^2 + 2a_2^2) + a_1^2 - 4a_1b + 4b^2 + a_2^2$

$= 3a_1^2 + 3a_2^2 + 6b^2$

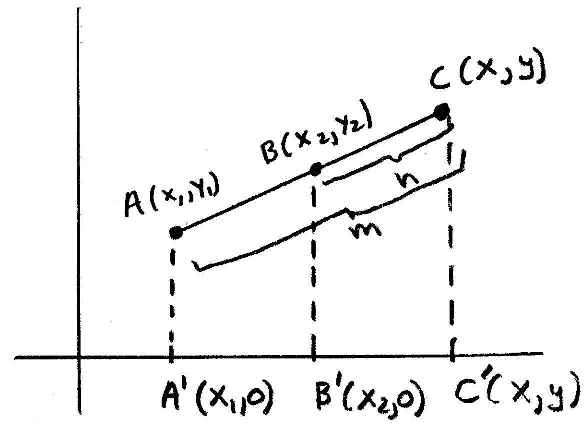
$= 3[a_1^2 + a_2^2 + 2b^2]$

$= 3(AD^2 + 2BD^2)$

$= RHS$

$\square$

[6]



By geometry, we know

$$\frac{AC}{BC} = \frac{A'C'}{B'C'}$$

Since C' EXT. divides $A'B'$,

$$x = \frac{mx_2 - nx_1}{m-n} .$$

The coordinate y of C is obtained similarly, thus

$$x = \frac{mx_2 - nx_1}{m-n}, \quad \frac{my_2 - ny_1}{m-n} = y$$

[7] $m/n = 3/2$. Get both internal and external

[7.1] $A(2,4)$, $B(5,1)$

<u>INT</u> $\quad x = \dfrac{3(5)+2(2)}{3+2} = \dfrac{19}{5}$

$$y = \frac{3(1)+2(4)}{5} = \frac{11}{5}$$

<u>EXT</u> $\quad x = \dfrac{3(5)-2(2)}{3-2} = \dfrac{11}{1} = 11$

$$y = \frac{3(1)-2(4)}{3-2} = -5$$

→

P 131, ctd

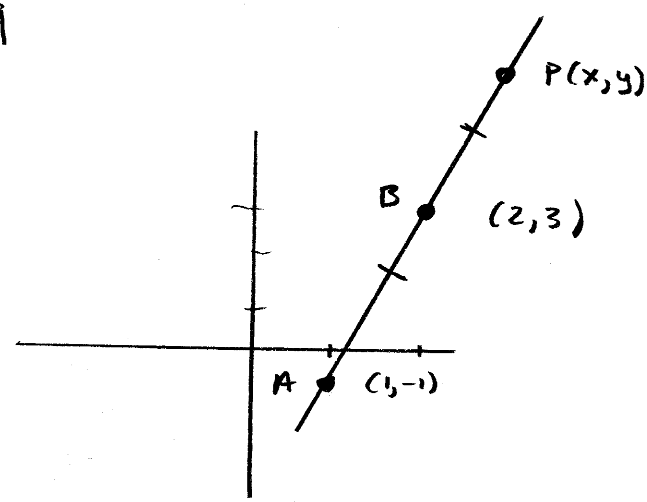
[7.2] INT $x = \dfrac{3(4)+2(-2)}{5} = \dfrac{8}{5}$

$y = \dfrac{3(-1)+2(3)}{5} = \dfrac{3}{5}$

EXT $x = \dfrac{3(4)-2(-2)}{3-2} = 16$

$y = \dfrac{3(-1)-2(3)}{3-2} = -9$

<u>P 132</u>



[8] P externally divides AB in ratio 1:1. I.e. B is the midpoint of AP. Thus,

$2 = \dfrac{1+x}{2}$, $3 = \dfrac{-1+y}{2}$

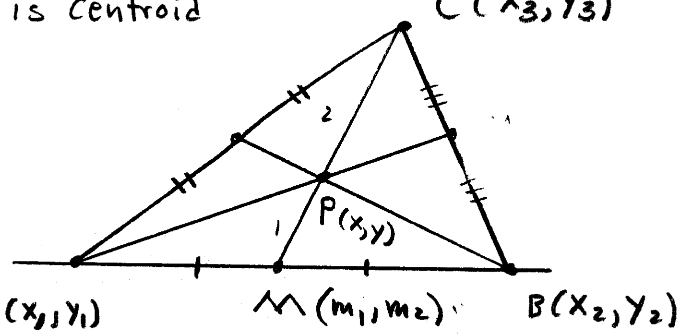
$4-1 = x$, $6+1 = y$

$\boxed{x = 3,\ y = 7}$

P 132, ctd

[9]

P(x,y) is centroid



P divides MC int in ratio 1:2, so

$$m_1 = \frac{x_1 + x_2}{2}$$

$$x = \frac{2m_1 + x_3}{2+1}$$

$$= \frac{(2)\left(\frac{x_1+x_2}{2}\right) + x_3}{3}$$

$$\therefore \quad x = \frac{x_1 + x_2 + x_3}{3}$$

similar for y

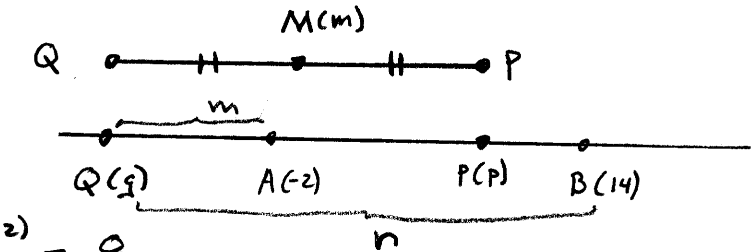
GEOM medians intersect at point $\frac{2}{3}$ distance from vertex to opposite side

$\square$

P132 EXERCISE

[1] A(-2), B(14). P INT DIV AB $m/n = 5/3$.
Q EXT DIV AB $m/n = 7/11$.

Get M(x,y), M midpt of PQ.



For P, $p = \dfrac{5(14)+3(-2)}{8} = 8$

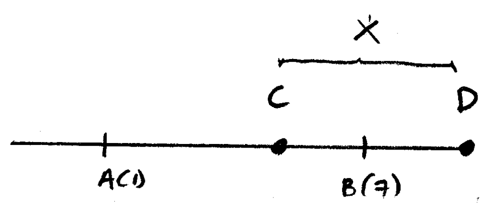
q, $q = \dfrac{7(14)-11(-2)}{7-11} = -30$

M: $m = \dfrac{-30+8}{2} = -11$

$\boxed{\therefore\ M(-11)}$

[2] A(1), B(7) C INT DIV AB $m/n = 4/3$
D EXT DIX AB $m/n = 4/3$

Get CD.



$c = \dfrac{4(7)+3(1)}{7} = \dfrac{31}{7}$

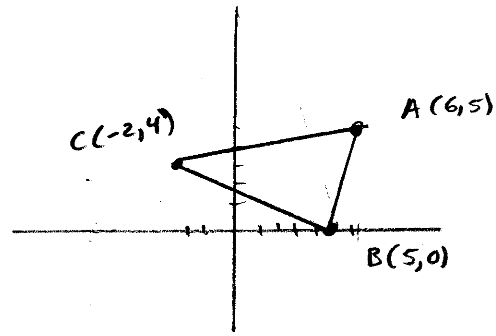
$d = \dfrac{(4)(7)-(3)(1)}{1} = 25$

$x = 25 - \frac{31}{7}$

$\boxed{x = \dfrac{144}{7}}$

P 132, Exercise, ctd

[3]



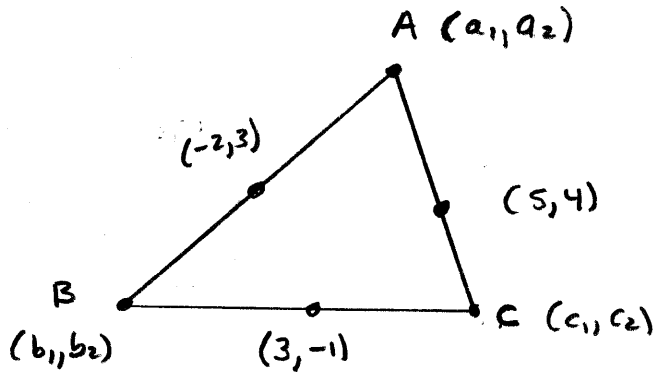
$$BC^2 = (5+2)^2 + (0-4)^2 = 65$$

$$AC^2 = (6+2)^2 + (5-4)^2 = 65$$

$$AB^2 = (6-5)^2 + (5-0)^2 = 26$$

ABC is isosceles.

[4]



$$\begin{bmatrix} \frac{1}{2}(a_1+b_1) = -2 \\ \frac{1}{2}(a_1+c_1) = 5 \\ \frac{1}{2}(b_1+c_1) = 3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 + 0 = -4 \\ a_1 + 0 + c_1 = 5 \\ 0 + b_1 + c_1 = 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 + b_1 = -4 \\ a_1 - b_1 = 2 \end{bmatrix} \Rightarrow 2a_1 = -2 \Rightarrow \begin{matrix} a_1 = -1 \\ b_1 = -3 \\ c_1 = 6 \end{matrix}$$

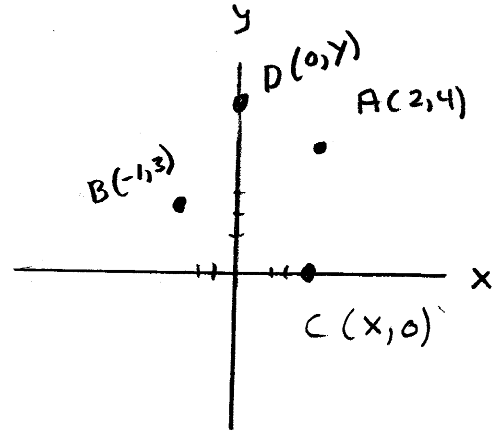
AND

$$\begin{bmatrix} a_2 + b_2 + 0 = 6 \\ a_2 + 0 + c_2 = 8 \\ 0 + b_2 + c_2 = -2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_2 + b_2 = 6 \\ a_2 - b_2 = 12 \end{bmatrix} \Rightarrow \begin{matrix} a_2 = 18 \\ b_2 = -12 \\ c_2 = -10 \end{matrix}$$

$$\boxed{\therefore \quad A(-1, 18),\ B(-3, -12),\ C(6, -10)}$$

P 132, ctd

[5]



$$AC^2 = (x-2)^2 + (0-4)^2$$
$$= x^2 - 4x + 4 + 16$$
$$= x^2 - 4x + 20$$

$$BC^2 = (x+1)^2 + (0-3)^2$$
$$= x^2 + 2x + 1 + 9$$
$$= x^2 + 2x + 10$$

$$AC^2 = BC^2$$
$$\equiv \quad x^2 - 4x + 20 = x^2 + 2x + 10$$
$$6x - 10 = 0$$
$$x = \frac{5}{3}$$

$\therefore \; C\left(\frac{5}{3}, 0\right)$

$$AD^2 = (0-2)^2 + (y-4)^2$$
$$= y^2 - 8y + 20$$
$$BD^2 = (0+1)^2 + (y-3)^2$$
$$= y^2 - 6y + 10$$

$$AD^2 = BD^2$$
$$\equiv \quad y^2 - 8y + 20 = y^2 - 6y + 10$$
$$2y - 10 = 0$$
$$\boxed{y = 5}$$

$\therefore \; D(0, 5)$

P147

[1.1] $y-5=-3(x-2)$

$$y=-3x+6+5$$

$$y=-3x+11$$

[1.2] $y-3=m(x+5)$

$$m=\frac{-1-3}{2+5}=-\frac{4}{7}$$

$$y-3=-\frac{4}{7}(x+5)$$

$$7y-21=-4x-20$$

$$4x+7y-1=0$$

[1.3] $y+4=2(x-3)$

$$y+4=2x-6$$

$$2x-y-10=0$$

[1.4] $y-1=m(x+1)$

$$x+3y+4=0$$

$$3y=-x-4$$

$$m_1=-\frac{1}{3}$$

$$mm_1=-1$$

$$m(-1/3)=-1$$

$$m=3$$

$$y-1=3(x+1)$$

$$y-1=3x+3$$

$$3x-y-4=0$$

[2] $\begin{bmatrix} 2x - y - 1 = 0 \\ 3x + 2y - 2 = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4x - 2y - 2 = 0 \\ 3x + 2y - 2 = 0 \end{bmatrix} \Rightarrow 7x - 4 = 0$

$\Rightarrow \boxed{x = \frac{4}{7}}$

$2\left(\frac{4}{7}\right) - y - 1 = 0$

$y = \frac{8}{7} - \frac{7}{7} = \frac{1}{7}$ $\boxed{y = \frac{1}{7}}$

$A(0,0)$, $B\left(\frac{4}{7}, \frac{1}{7}\right)$

$l: \quad y - 0 = m(x - 0)$

$m = \dfrac{1/7}{4/7} = \frac{1}{7} \cdot \frac{7}{4} = \frac{1}{4}$

$\boxed{y = \frac{1}{4}x}$

[3] $l_1: \; 3x + 4y - 5 = 0$
$l_2: \; 4x + ky - 4 = 0$

$m_1 = -\frac{3}{4}$

$m_2 = -\frac{4}{k}$

$m_1 m_2 = -1 \iff -\frac{3}{4} \cdot \frac{-4}{k} = -1$

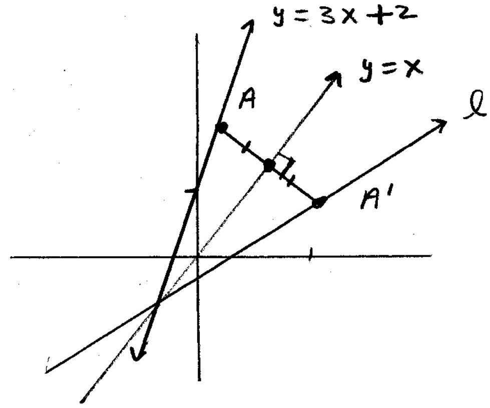
$\iff \frac{3}{k} = -1$

$\iff k = -3$

$\therefore \; k = -3$

[4]

Let l be the line symmetric to $y=3x+2$ w.r.t. $y=x$.



Choose Points A, B on $y=3x+2$, for example $A(0,2)$, $B(1,5)$

Then $A'(2,0)$ and $B'(5,1)$ are on l. Thus,

$$l: \quad y-0=m(x-2)$$

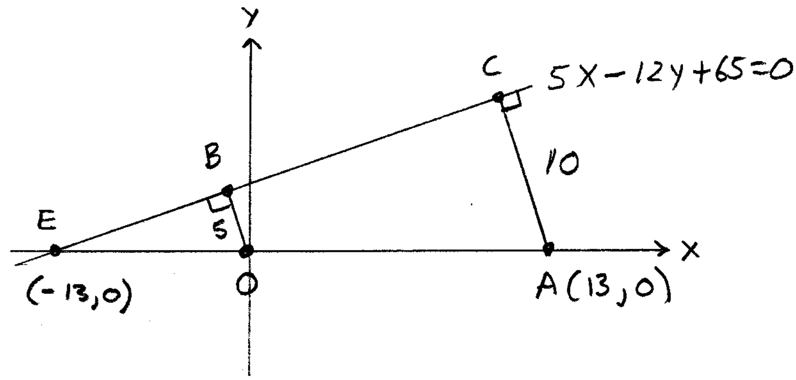
$$m=\frac{1-0}{5-2}=\frac{1}{3}$$

So, $l: \; y=\frac{1}{3}(x-2)$

$$3y=x-2$$

$$\therefore \; \boxed{l: \; x-3y-2=0}$$

[5]



$$A_{OACB} = A_{\Delta EAC} - A_{\Delta EOB}$$

coords of E:

$$y=0 \Rightarrow 5x - 12(0) + 65 = 0$$
$$\Rightarrow x = -13$$

$E(-13,0)$

$$A_{EAC} = \frac{1}{2}(\overline{AC})(\overline{EC})$$

$$\overline{AC} = \frac{|5(13) - 12(0) + 65|}{\sqrt{25+144}} = 10 \Rightarrow \boxed{\overline{AC} = 10}$$

$$\overline{EC} = \sqrt{26^2 - 10^2} = \sqrt{576} = 24 \Rightarrow \boxed{\overline{EC} = 24}$$

$$A_{EAC} = \frac{1}{2}(10)(24) = 120 \Rightarrow \boxed{A_{\Delta EAC} = 120}$$

$$A_{EOB} = \frac{1}{2}(\overline{EB})(\overline{OB})$$

$$\overline{OB} = \frac{|5(0) - 12(0) + 65}{\sqrt{25+144}} = \frac{65}{13} \Rightarrow \boxed{\overline{OB} = 5}$$

$$\overline{EB} = \sqrt{13^2 - 25} = \sqrt{144} = 12 \Rightarrow \boxed{\overline{EB} = 12}$$

$$A_{EOB} = \frac{1}{2}(12)(5) = 30 \quad \boxed{A_{\Delta EOB} = 30}$$

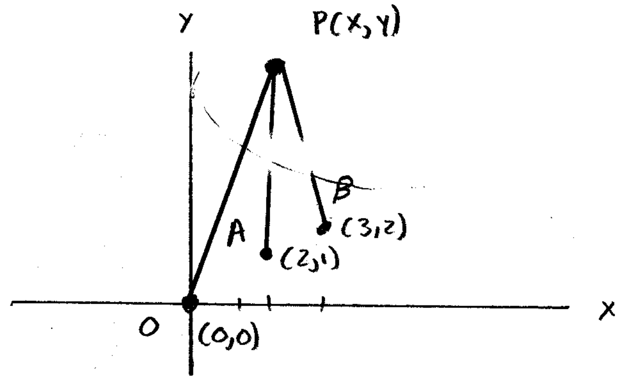
Then,

$$A_{OACB} = 120 - 30$$
$$= 90$$

$$\therefore A_{OACB} = 90$$

[6] $O(0,0)$, $A(2,1)$, $B(3,2)$. GET LOCUS of P, s.t.

$$\overline{PO}^2 + \overline{PA}^2 = 2\,\overline{PB}^2$$



$$\overline{PO}^2 = (x-0)^2 + (y-0)^2$$
$$= x^2 + y^2$$

$$\overline{PA}^2 = (x-2)^2 + (y-1)^2$$
$$= x^2 - 4x + 4 + y^2 - 2y + 1$$

$$\overline{PB}^2 = (x-3)^2 + (y-2)^2$$
$$= x^2 - 6x + 9 + y^2 - 4y + 4$$

Then, $\overline{PO}^2 + \overline{PA}^2 = 2\,\overline{PB}^2$

$$\Rightarrow \quad \cancel{x^2} + \cancel{y^2} + \cancel{x^2} - \cancel{4x} + \cancel{4} + \cancel{y^2} - \cancel{2y} + \cancel{1} = \cancel{2x^2} - 12\cancel{x} + 1\cancel{8} + \cancel{2y^2} - 8\cancel{y} + \cancel{8}$$

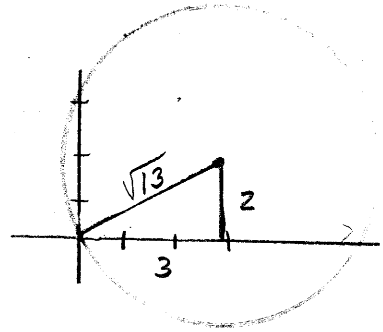
$$\Rightarrow \quad 8x - 21 + 6y = 0$$

$$\Rightarrow \quad 6y = -8x + 21$$
$$y = -\frac{8}{6}x + \frac{21}{6}$$
$$y = -\frac{4}{3}x + \frac{7}{2}$$

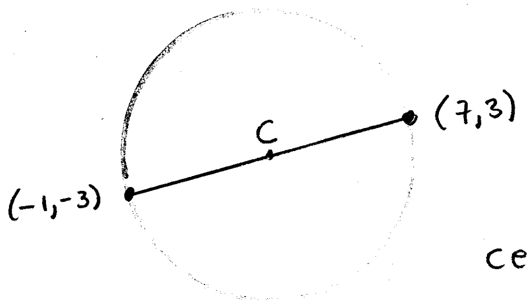
P149

[1.1] $(x+2)^2+(y-1)^2=3$

[1.2] $(x-3)^2+(y+2)^2=13$



[1.3]



center: $x=\frac{7-1}{2}$, $y=\frac{3-3}{2}$

$C\ (3,0)$

$$\text{radius} = \frac{1}{2}\sqrt{(7+1)^2+(3+3)^2}$$
$$= \frac{1}{2}\sqrt{64+36}$$
$$= 5$$

$\therefore\ (x-3)^2+y^2=25$

[1.4] $(x-h)^2+y^2=r^2$ is eqn circle center on x-axis, $k=0$.

since $(3,\sqrt{3})$ and $(2,-2)$ on circle;

(1) $(3-h)^2+(\sqrt{3})^2=r^2$
$9-6h+h^2+3=r^2$
$h^2-6h+12=r^2$

and (2) $(2-h)^2+(-2)^2=r^2$
$4-4h+h^2+4=r^2$
$h^2-4h+8=r^2$

then $h^2-6h+12=h^2-4h+8$
$2h=4$
$\boxed{h=2}$

and $(2-2)^2+(-2)^2=r^2 \Rightarrow 0+4=r^2 \Rightarrow r^2=4$

$\therefore\ (x-2)^2+y^2=4$

149, ctd

[2.1] $x^2+y^2-6x+4y+4=0$

$$(x^2-6x+9)+(y^2+4y+4) = -4+9+4$$

$$(x-3)^2+(y+2)^2 = 9$$

$\therefore$ Circle radius 3 center $(3,-2)$

[2.2] $3x^2+3y^2-2x-3y+1=0$

$$x^2+y^2-\frac{2}{3}x-y+\frac{1}{3}=0$$

$$\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+\left(y^2-y+\frac{1}{4}\right) = -\frac{1}{3}+\frac{1}{9}+\frac{1}{4}$$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{36}$$

[3]

$$\begin{bmatrix} 36+36-6A+6B+C=0 \\ 4+64-2A+8B+C=0 \\ 1+1+A-B+C=0 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 6A-6B-C = 72 \\ 2A-8B-C = 68 \\ A-B+C = -2 \end{bmatrix}$$

$\Rightarrow$ $A=4$, $B=-6$, $C=-12$

$\therefore$ $x^2+y^2+4x-6y-12=0$

[4] $A(-2,0)$, $B(3,0)$. $AP : PB = 3:2$, $P(x,y)$.

$$\frac{AP}{BP} = \frac{3}{2} \Rightarrow \boxed{AP = \frac{3}{2} BP}$$

$$AP^2 = (x+2)^2 + (y-0)^2$$
$$\boxed{= x^2+y^2+4x+4}$$
$$BP^2 = (3-x)^2 + (0-y)^2$$
$$\boxed{= x^2+y^2-6x+9}$$

Then, $AP^2 = \frac{9}{4} BP^2$

$$x^2+y^2+4x+4 = \frac{9}{4}\left[x^2+y^2-6x+9\right]$$
$$4x^2+4y^2+16x+16 = 9x^2+9y^2-54x+81$$
$$\therefore \quad 5x^2+5y^2-70x+65=0$$

p. 151

[5] $A(1,0)$, $B(-2,3)$, $C(-5,-3)$. $P(x,y) \ni AP^2 = BP^2 + CP^2$

$$AP^2 = (x-1)^2+(y-0)^2 = x^2+y^2-2x+1$$
$$BP^2 = (x+2)^2+(y-3)^2 = x^2+y^2+4x-6y+13$$
$$CP^2 = (x+5)^2+(y+3)^2 = x^2+y^2+10x+6y+34$$

$$AP^2 = BP^2 + CP^2$$
$$\equiv \quad x^2+y^2-2x+1 = 2x^2+2y^2+14x+47$$
$$\equiv \quad x^2+y^2+16x+46=0$$
$$\equiv \quad (x+8)^2+y^2 = -46+64$$
$$\equiv \quad (x+8)^2+y^2 = 18$$

$\therefore$ Circle radius $3\sqrt{2}$ center $(-8,0)$

$$(x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2=2k^2$$

$$(x-x_1)^2+(x-x_2)^2+(y-y_1)^2+(y-y_2)^2=2k^2$$

$$2x^2-2xx_1+x_1^2-2xx_2+x_2^2+2y^2-2yy_1+y_1^2-2yy_2+y_2^2=2k^2$$

$$2x^2-2xx_1-2xx_2+2y^2-2yy_1-2yy_2=2k^2-(x_1^2+y_2^2+x_2^2+y_1^2)$$

$$x^2-xx_1-xx_2+y^2-yy_1-yy_2=k^2-\frac{1}{2}(x_1^2+y_2^2+x_2^2+y_1^2)$$

$$x^2-(x_1+x_2)x+y^2-(y_1+y_2)y=k^2-\frac{1}{2}(x_1^2+y_2^2+x_2^2+y_1^2)$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=k^2-\frac{1}{2}(x_1^2+y_2^2+x_2^2+y_1^2)+\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=$$
$$\frac{4k^2}{4}-\frac{x_1^2}{2}-\frac{x_2^2}{2}-\frac{y_1^2}{2}-\frac{y_2^2}{2}+\frac{x_1^2}{4}+\frac{2x_1x_2}{4}+\frac{x_2^2}{4}+\frac{y_1^2}{4}+\frac{2y_1y_2}{4}+\frac{y_2^2}{4}$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\frac{1}{4}(4k^2-x_1^2+2x_1x_2-x_2^2-y_1^2+2y_1y_2-y_2^2)$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\frac{1}{4}(4k^2-(x_1^2-2x_1x_2+x_2^2)-(y_1^2-2y_1y_2+y_2^2)$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\frac{1}{4}\left(4k^2-\left((x_1^2-x_2^2)^2+(y_1^2-y_2^2)^2\right)\right)$$

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\frac{1}{4}(4k^2-AB^2)$$

$$r^2=\frac{1}{4}(4k^2-AB^2)$$

$$r=\frac{1}{2}\sqrt{4k^2-AB^2}$$

$r=\frac{1}{2}\sqrt{4k^2-AB^2}$, which is a real number when $AB<2k$

∴ Circle: center the midpoint of AB, radius $\frac{1}{2}\sqrt{4k^2-AB^2}$ whenever $AB<2k$.

Particular examples that illustrate the above

■ Suppose $A(-3, 0)$, $B(3, 0)$, then

$AB = 6, \ \text{so } k > 3$

Choose $k = 4$. Then

$$r = \frac{1}{2}\sqrt{4\,k^2 - AB^2}$$

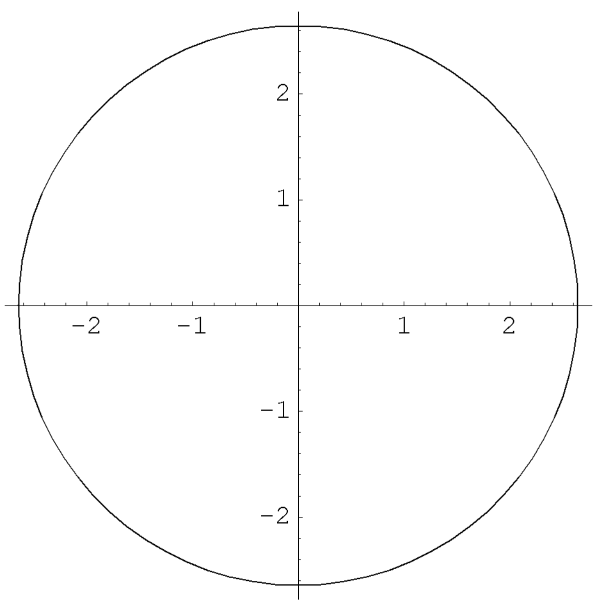
$$r = \frac{1}{2}\sqrt{4 * 16 - 36}$$

$$r = \left(\sqrt{7}\right) \simeq 2.6$$

So $C(0, 0)$, $r = \sqrt{7}$

So $C(0, 0)$, $r = \sqrt{7} \approx 2.6$

In[58]:= `ImplicitPlot[(x + 3)^2 + y^2 + (x - 3)^2 + y^2 == 2 * 4 * 4, {x, -5, 5}];`



■ Suppose $A(5, 2)$, $B(7, 11)$, then

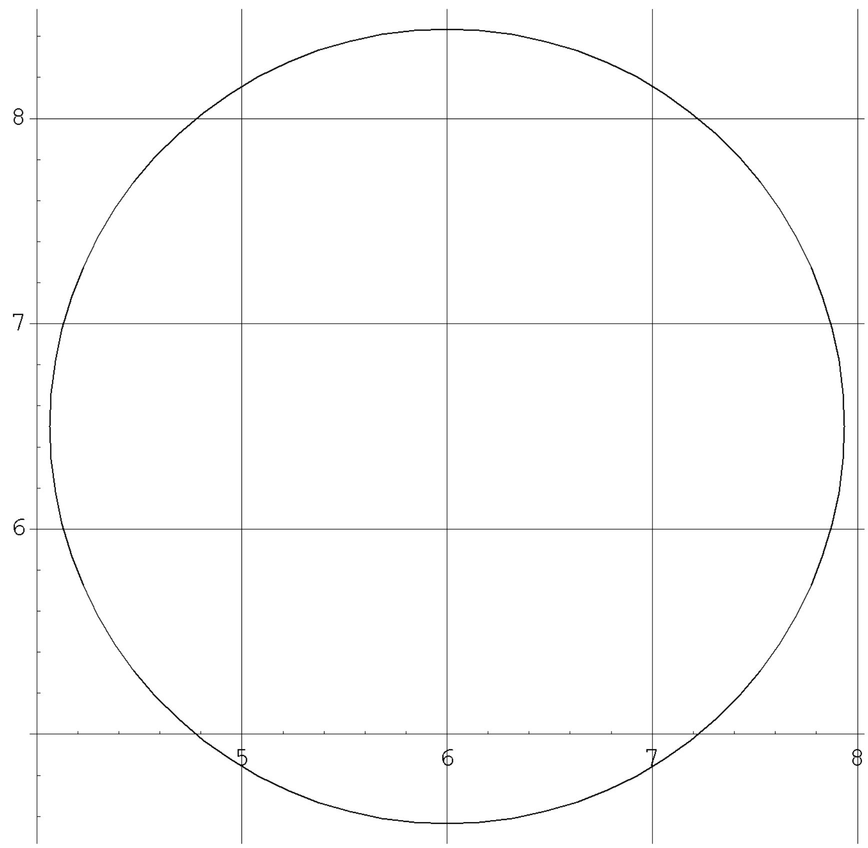
$$AB = \sqrt{(7-5)^2 + (11-2)^2} = \sqrt{85}, \ \text{so } k > \frac{\sqrt{85}}{2} \approx 4.6$$

Choose $k = 5$. Then

$$r = \frac{1}{2}\sqrt{4\,k^2 - AB^2} = \frac{1}{2}\sqrt{4\,(5^2) - 85} = \frac{\sqrt{15}}{2}$$

So $C(\frac{7+5}{2},\ \frac{11+2}{2}),\ r = \frac{\sqrt{15}}{2} \iff C(6,\ \frac{13}{2}),\ r = \frac{\sqrt{15}}{2} \approx 1.9$

In[72]:=
```
ImplicitPlot[(x - 5)^2 + (y - 2)^2 + (x - 7)^2 + (y - 11)^2 == 2 (25),
  {x, -15, 15}, GridLines → Automatic];
```



P152

[1] $\begin{bmatrix} 2x - y - 2 = 0 \\ x^2 + y^2 - x + 4y = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y = 2x - 2 \\ x^2 + y^2 - x + 4y = 0 \end{bmatrix}$

Then,

$$x^2 + (2x-2)^2 - x + 4(2x-2) = 0$$

$$x^2 + 4x^2 - 8x + 4 - x + 8x - 8 = 0$$

$$5x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1 + 4(5)(4)}}{10}$$

$$= \frac{1 \pm \sqrt{81}}{10}$$

$$= \frac{1 \pm 9}{10}$$

$$y = 2\left(-4/5\right) - 2$$

$$= -\frac{8}{5} - 2$$

$$= -\frac{16}{10} - \frac{20}{10}$$

$x = 1$ or $x = -\frac{4}{5}$

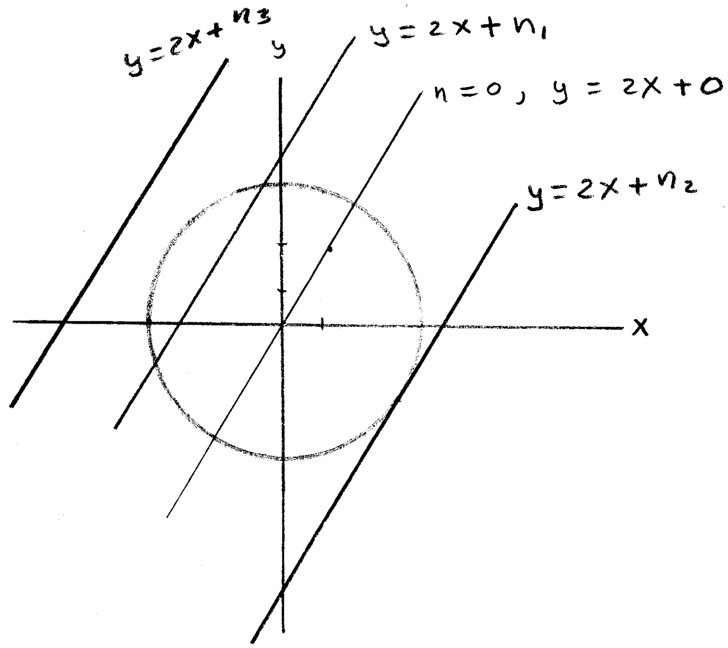
$y = 0$ or $y = -\frac{18}{5}$

$\therefore$ $(1, 0)$ and $\left(-\frac{4}{5}, -\frac{18}{5}\right)$

P 152 DEM 2

$x^2 + y^2 = 1$

$y = 2x + n$



Number of points in common $\{0,1,2\}$ is the same as the quantity of solns of

$$\begin{bmatrix} x^2 + y^2 = 1 \\ y = 2x + n \end{bmatrix}$$

$$\equiv \quad x^2 + (2x+n)^2 = 1$$

$$x^2 + 4x^2 + 4nx + n^2 = 1$$

$$5x^2 + 4nx + n^2 = 1$$

$$5x^2 + 4nx + n^2 - 1 = 0$$

} Qty solns given by discriminant. Could be $\{0,1,2\}$

$$D = 16n^2 - 4(5)(n^2-1)$$

$$= 16n^2 - 20n^2 + 20$$

$$D = -4n^2 + 20$$

$$\boxed{\frac{D}{4} = 5 - n^2}$$

$D > 0 \Rightarrow -\sqrt{5} < n < \sqrt{5}$ two real solns

$D = 0 \Rightarrow n = \pm\sqrt{5}$ one soln multiplicity two

$D < 0 \Rightarrow n < -\sqrt{5}$ or $n > \sqrt{5}$ no real solns

$\therefore$ For $-\sqrt{5} < n < \sqrt{5}$, two points common

$n = \pm\sqrt{5}$, one point common $\therefore$ tangent

$n < -5,\ n > 5$, No Points common

p153,

[2] $y = x + n$

$x^2 + y^2 = 4$, where $y = x + n$ tangent to $x^2 + y^2 = 4$

Soln

$$x^2 + (x+n)^2 = 4$$

$$x^2 + x^2 + 2nx + n^2 = 4$$

$$2x^2 + 2nx + n^2 - 4 = 0 \qquad (i)$$

Tangent if (i) has one Soln multiplicity 2; I.e. $D = 0$

So,

$$D = 4n^2 - (4)(2)(n^2 - 4)$$

$$= 4n^2 - 8n^2 + 32$$

$$= -4n^2 + 32$$

$D = 0$

$$-4n^2 + 32 = 0$$

$$n^2 - 8 = 0$$

$$n = \pm\sqrt{8}$$

$$n = \pm 2\sqrt{2}$$

$\therefore$ Tangent when $n = 2\sqrt{2}$ or $n = -2\sqrt{2}$

P 153 J10

[3]

$$\begin{bmatrix} y = mx - 2 \\ x^2 + y^2 = 1 \end{bmatrix}$$

$$\Rightarrow \quad x^2 + (mx-2)^2 = 1$$

$$x^2 + m^2x^2 - 4mx + 4 = 1$$

$$(m^2+1)x^2 - 4mx + 3 = 0$$

$$D = [-4m]^2 - (4)(m^2+1)(3)$$

$$= 16m^2 - 12m^2 - 12$$

$$= 4m^2 - 12$$

Set $D > 0$

$$4m^2 - 12 > 0$$

$$m^2 - 3 > 0$$

$$(m - \sqrt{3})(m + \sqrt{3}) > 0$$

		zero		zero	
$m^2 - 3$	+		−		+
$m + \sqrt{3}$	−		+		+
$m - \sqrt{3}$	−		−		+
		$-\sqrt{3}$		$\sqrt{3}$	

so $D > 0$ when $m < -\sqrt{3}$ or $m > \sqrt{3}$

$\boxed{\therefore \text{ Two common points for } m < -\sqrt{3} \text{ or } m > \sqrt{3}}$ (a)

$\boxed{\therefore \text{ Tangent when } m = -\sqrt{3} \text{ or } m = \sqrt{3}}$ (b)

P154

[4] $x^2+y^2=25$. Lines tangent at (a) $(3,4)$, (b) $(0,5)$, (c) $(-4,3)$

use $x_0 x + y_0 x = r^2$

$x_0 x + y_0 y = 25$

(a) $3x+4y=25$

(b) $5y = 25 \equiv y=5$

(c) $-4x+3y=25$

$4x-3y=-25$

P155 [1]

$x^2+y^2=r^2$

$y = mx+b$

Since given slope m, need only find b.

$x^2+[mx+b]^2=r^2$

$x^2+m^2x^2+2mbx+b^2-r^2=0$

$(m^2+1)x^2+2mb\,x+(b^2-r^2)=0$

If line tangent to circle, the one SOLN mult 2,

$[2mb]^2-(4)(m^2+1)(b^2-r^2)=0$

$4m^2b^2-4[m^2b^2+b^2-m^2r^2-r^2]=0$

$m^2b^2-m^2b^2-b^2+m^2r^2+r^2=0$

$-b^2+m^2r^2+r^2=0$

$b^2=m^2r^2+r^2$

$b^2=(m^2+1)r^2$

$b=\pm r\sqrt{m^2+1}$

$\therefore\ y=mx\pm r\sqrt{m^2+1}$

[3]

$$\begin{bmatrix} y = mx - 2 \\ x^2 + y^2 = 1 \end{bmatrix}$$

$$\Rightarrow \quad x^2 + (mx-2)^2 = 1$$

$$x^2 + m^2x^2 - 4mx + 4 = 1$$

$$(m^2+1)x^2 - 4mx + 3 = 0$$

$$D = [-4m]^2 - (4)(m^2+1)(3)$$

$$= 16m^2 - 12m^2 - 12$$

$$= 4m^2 - 12$$

Set $D > 0$

$$4m^2 - 12 > 0$$

$$m^2 - 3 > 0$$

$$(m-\sqrt{3})(m+\sqrt{3}) > 0$$

		zero		zero	
m^2-3	$+$		$-$		$+$
$m+\sqrt{3}$	$-$		$+$		$+$
$m-\sqrt{3}$	$-$		$-$		$+$
		$-\sqrt{3}$		$\sqrt{3}$	

so $D > 0$ when $m < -\sqrt{3}$ or $m > \sqrt{3}$

$\boxed{\therefore \text{ Two common points for } m < -\sqrt{3} \text{ or } m > \sqrt{3}}$ (a)

$\boxed{\therefore \text{ Tangent when } m = -\sqrt{3} \text{ or } m = \sqrt{3}}$ (b)

P154

[4] $x^2+y^2=25$. Lines tangent at (a) $(3,4)$, (b) $(0,5)$, (c) $(-4,3)$

use $x_0 x + y_0 x = r^2$

$x_0 x + y_0 y = 25$

(a) $3x+4y=25$

(b) $5y=25 \equiv y=5$

(c) $-4x+3y=25$

$4x-3y=-25$

P155 [1]

$x^2+y^2=r^2$

$y=mx+b$

Since given slope m, need only find b.

$x^2+[mx+b]^2=r^2$

$x^2+m^2x^2+2mbx+b^2-r^2=0$

$(m^2+1)x^2+2mb\,x+(b^2-r^2)=0$

If line tangent to circle, the one SOLN mult 2,

$[2mb]^2-(4)(m^2+1)(b^2-r^2)=0$

$4m^2b^2-4[m^2b^2+b^2-m^2r^2-r^2]=0$

$m^2b^2-m^2b^2-b^2+m^2r^2+r^2=0$

$-b^2+m^2r^2+r^2=0$

$b^2=m^2r^2+r^2$

$b^2=(m^2+1)r^2$

$b=\pm r\sqrt{m^2+1}$

$\therefore\ y=mx\pm r\sqrt{m^2+1}$

P156

[6] Let l with slope m be tangent to $x^2+y^2=9$ at (x_0, y_0).
since l perp to $3x+y=3$,

$m = -\frac{1}{3}$, so

$\frac{y_0}{x_0} = -\frac{1}{3}$.

But $x_0^2 + y_0^2 = 9$

$x_0^2 + \left[\frac{-x_0}{3}\right]^2 = 9$

$10x_0^2 = 81$

$x_0 = \pm \frac{9}{\sqrt{10}}$

$x_0 = \frac{9}{\sqrt{10}}$

$y_0 = -\frac{1}{3}\left(\frac{9}{\sqrt{10}}\right)$

$= -\frac{3}{\sqrt{10}}$

$x_0 = \frac{-9}{\sqrt{10}}$

$y_0 = -\frac{1}{3}\left(-\frac{9}{\sqrt{10}}\right)$

$= \frac{3}{\sqrt{10}}$

$\therefore$

$l_1 : \frac{9}{\sqrt{10}}x - \frac{3}{\sqrt{10}}y = 9$, $l_2 : -\frac{9}{\sqrt{10}}x + \frac{3}{\sqrt{10}}y = 9$

$$\boxed{\begin{array}{l} l_1 \quad 9x - 3y = 9\sqrt{10} \\ l_2 \quad 9x - 3y = -9\sqrt{10} \end{array}}$$

[7] Let (x_0, y_0) point of tangency. Then line tangent is

$$x_0 x + y_0 y = 4$$

Since $(5, 0)$ is on this line,

$$5x_0 + 0y_0 = 4 \implies \boxed{x_0 = \frac{4}{5}}$$

But (x_0, y_0) on circle, so

$$x_0^2 + y_0^2 = 4$$

$$\left(\frac{4}{5}\right)^2 + y_0^2 = 4$$

$$y_0^2 = 4 - \frac{16}{25}$$

$$y_0 = \pm\sqrt{\frac{84}{25}}$$

$$= \pm \frac{2}{5}\sqrt{21}$$

$$\boxed{\begin{array}{l} l_1: \ \frac{4}{5}x + \frac{2\sqrt{21}}{5}y = 4 \\ l_2: \ \frac{4}{5}x - \frac{2\sqrt{21}}{5}y = 4 \end{array}}$$

P157

[1.1] Right 1, up 2

[1.2] Right 3, down 3

[2] $x^2+y^2-2x+4y-4=0$ Right 2, down 1

$$(x-2)^2+(y+1)^2-2(x-2)+4(y+1)-4=0$$

$$x^2-4x+4+y^2+2y+2-2x+4+4y+4-4=0$$

$$x^2-6x+y^2+6y+10=0$$

$$\therefore\ x^2+y^2-6x+6y+10=0$$

P158

[3] $y=4x+3$ Right 4, Down 3

$$y+3=4(x-4)+3$$

$$y+3=4x-16+3$$

$$\boxed{y=4x-16}$$

[4] $y=3x-1$, Right a, up $2a$. l' includes $(0,0)$

$$y-2a=3(x-a)-1$$

$$y-2a=3x-3a-1$$

$$a=3x-y-1$$

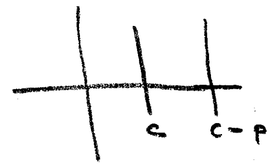
$$(x,y)=(0,0)$$

$$\Rightarrow a=-1$$

$$\boxed{\therefore\ a=-1}$$

P159

[5] $x = c$, p right, q up



$$x - p = c$$

$$\boxed{x = p + c}$$

[6] $ax + by + c = 0$, p right, q up

$$a(x-p) + b(y-q) + c = 0$$

$$ax + by - ap - bq + c = 0$$

[7.1] 2R, 1 down

$$(x-2) + 2(y+1) - 3 = 0$$

$$x - 2 + 2y + 2 - 3 = 0$$

$$x + 2y - 3 = 0$$

[7.2] $(x-2)^2 + (y+1)^2 = 1$

[7.3] $(x-2)^2 - (y+1) = 0$

$$x^2 - 4x + 4 - y - 1 = 0$$

$$x^2 - 4x - y + 3 = 0$$

$$(x-2)^2 - y + 3 - 4 = 0$$

$$\boxed{(x-2)^2 - y = 1}$$

P160

[1.1] $C(2,-1)$, $P(-1,3)$

$$r^2 = (-1-2)^2 + (3+1)^2$$
$$= 9 + 16$$
$$= 25$$

$$\therefore \ (x-2)^2 + (y+1)^2 = 25$$

[1.2] $(-4,1)$, $(4,5)$ center on $2x - y = 1$

Let (x_0, y_0) be center of circle.

(1) $r^2 = (x_0+4)^2 + (y_0-1)^2$

(2) $r^2 = (x_0-4)^2 + (y_0-5)^2$

$$(1,2) \Rightarrow \cancel{x_0^2} + 8x_0 + \cancel{16} + \cancel{y_0^2} - 2y_0 + 1 = \cancel{x_0^2} - 8x_0 + \cancel{16} + \cancel{y_0^2} - 10y_0 + 25$$

$$16x_0 + 8y_0 = 24$$

$$\boxed{2x_0 + y_0 = 3}$$

But x_0, y_0 also on $2x - y = 1$, so

$$\begin{bmatrix} 2x_0 + y_0 = 3 \\ 2x_0 - y_0 = 1 \end{bmatrix} \Rightarrow 2y_0 = 2 \Rightarrow \boxed{y_0 = 1}$$

$$2x_0 + 1 = 3$$

$$\boxed{x_0 = 1}$$

Center $(1,1)$

$$r^2 = (1-4)^2 + (1-5)^2$$
$$= 9 + 16$$
$$= 25$$

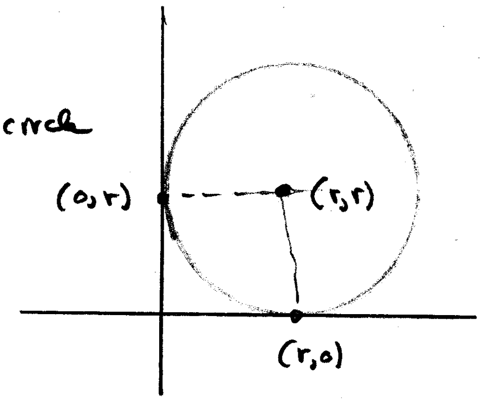
$$\boxed{\therefore \ (x-1)^2 + (y-1)^2 = 25}$$

[1.3] (1,2) and tangent to both axes

P (1,2). Tangent to x=0 and y=0

circle $(x-r)^2+(y-r)^2=r^2$

$(1-r)^2+(2-r)^2=r^2$ (1,2) on circle



$1-2r+r^2+4-4r+r^2=r^2$

$5-6r+r^2=0$

$r^2-6r+5=0$

$(r-1)(r-5)=0$

$r=1$ or $r=5$

$\therefore$ circle $\boxed{\begin{array}{l}(x-1)^2+(y-1)^2=1 \\ \text{or} \\ (x-5)^2+(y-5)^2=25\end{array}}$

P160 ctd

[2] $A(0,0)$, $B(4,2)$, $C(5,1)$

$AP^2 + BP^2 + CP^2 = 49$

$\equiv \quad x^2 + y^2 + (x-4)^2 + (y-2)^2 + (x-5)^2 + (y-1)^2 = 49$

$\equiv \quad 3x^2 + 3y^2 - 8x - 10x - 4y - 2y + 16 + 4 + 25 + 1 = 49$

$\equiv \quad 3x^2 - 18x + 3y^2 - 6y = 3$

$x^2 - 6x + y^2 - 2y = 1$

$(x-3)^2 + (y-1)^2 = 1 + 9 + 1$

$(x-3)^2 + (y-1)^2 = 11$

$\therefore$ Circle Center $(3, 1)$, radius $\sqrt{11}$

[3] $(5-2)^2 + (4-3)^2 = 10$ ✓

Line tangent to $x^2+y^2=10$ at $(3,1)$

is $3x + y = 10$. So Tangent to $(x-2)^2 + (y-3)^2 = 10$

at $3(x-2) + (y-3) = 10$

$\equiv \quad 3x - 6 + y - 3 = 0$

$\equiv \quad 3x + y - 9 = 0$

P160, ctd

[4] EQN LINE $\quad x_0 x + y_0 y = 25$

(1,7) on l $\quad x_0 + 7y_0 = 25$

$$\boxed{x_0 = 25 - 7y_0}$$

$$x_0^2 + y_0^2 = 25$$

$$(25 - 7y_0)^2 + y_0^2 = 25$$

$$625 - 350y_0 + 49y_0^2 + y_0^2 = 25$$

$$50y_0^2 - 350y_0 + 625 = 25$$

$$50y_0^2 - 350y_0 + 600 = 0$$

$$y_0^2 - 7y_0 + 12 = 0$$

$$(y_0 - 3)(y_0 - 4) = 0$$

$y_0 = 3$

$x_0 = 25 - 21 = 3$

$(x_0, y_0) = (3, 3)$

$y_0 = 4$

$x_0 = 25 - 7(4)$

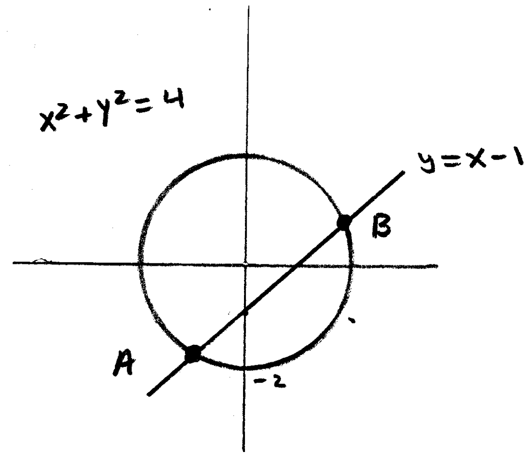
$= -3$

$(-3, 4)$

$$\therefore \boxed{\begin{array}{ll} l_1 & 3x + 3y = 25 \\ l_2 & -3x + 4y = 25 \end{array}}$$

p 160 ctd

[5]



$$\begin{bmatrix} y = x-1 \\ x^2+y^2=4 \end{bmatrix}$$

$$x^2+(x-1)^2=4$$

$$x^2+x^2-2x+1=4$$

$$2x^2-2x=3$$

$$2x^2-2x-3=0$$

$$x=\frac{2\pm\sqrt{4+24}}{4}$$

$$=\frac{2\pm2\sqrt{7}}{4}$$

$$=\frac{1\pm\sqrt{7}}{2}$$

$$A\left(\frac{1+\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right), B\left(\frac{1-\sqrt{7}}{2}, \frac{-1-\sqrt{7}}{2}\right)$$

$$AB^2=\left(\frac{1-\sqrt{7}}{2}-\frac{1+\sqrt{7}}{2}\right)^2+\left(\frac{-1-\sqrt{7}}{2}-\frac{-1+\sqrt{7}}{2}\right)^2$$

$$=(-\sqrt{7})^2+(-\sqrt{7})^2$$

$$=14$$

$$\therefore \boxed{AB=\sqrt{14}}$$

[6] $x^2 + [mx+2]^2 = 1$

$x^2 + m^2x^2 + 4mx + 4 = 1$

$(m^2+1)x^2 + 4mx + 3 = 0$

$D = (4m)^2 - 4(3)(m^2+1)$

$= 16m^2 - 12m^2 - 12$

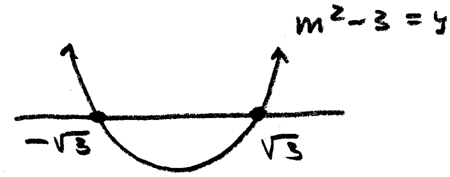
$= 4m^2 - 12$

set $D > 0$

$4m^2 - 12 > 0$

$m^2 - 3 > 0$

$(m-\sqrt{3})(m+\sqrt{3}) > 0$



$\therefore$ Line cuts circle twice for $m < -\sqrt{3}$ or $m > \sqrt{3}$

Line tangent when $m = \pm\sqrt{3}$.

P 160 ctd

[7] $x^2+y^2+Ax+By+C=0$

$$\begin{bmatrix} C=0 \\ 1+4+A+2B+C=0 \\ 16+9+4A+3B+C=0 \end{bmatrix}$$

$$\equiv \begin{bmatrix} A+2B=-5 \\ 4A+3B=-25 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 4A+8B=-20 \\ 4A+3B=-25 \end{bmatrix} \Rightarrow 5B=5$$

$\therefore B=1$

$A=-7$

$$\boxed{\therefore\ x^2+y^2-7x+y=0}$$

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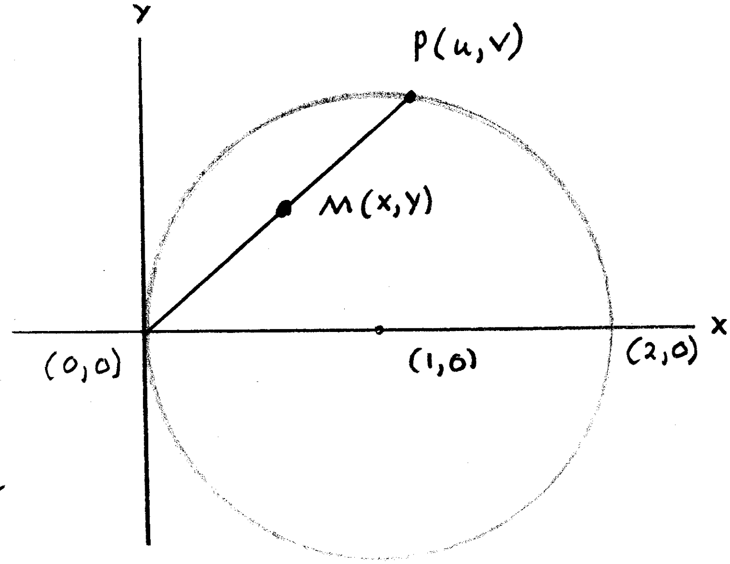
[8]



$x = \frac{u}{2}, \ y = \frac{v}{2}$

$(u-1)^2 + v^2 = 1$

$u^2 - 2u + 1 + v^2 = 1$

$4x^2 - 4x + 1 + 4y^2 = 1$

$x^2 + y^2 - x = 0$, circle

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

$\therefore$ Locus is circle center $(\frac{1}{2}, 0)$ radius $\frac{1}{2}$.